COMPLEX VARIABLE BOUNDARY INTEGRAL MODELING OF GROUNDWATER FLOW AND TRANSPORT

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Abstract. Solution of difficult groundwater flow and transport problems is made easier when the groundwater equipotentials are coupled with streamlines. This approach, embedded in a new formulation of the Complex Variable Boundary Element Method (CVBEM), is shown to readily solve groundwater flow and transport problems. Examples of this method are presented for recalcitrant problems related to flow under a dam, dipole transport, and regional groundwater flow. The presented technique solves the two-dimensional Laplace equation using Ordinary Least Squares (CVBEM/OLS). This strategy is used to determine unknown boundary values at nodes along the exterior boundary. Once all boundary values are known, the problem is reformulated in the complex potential plane to solve for the complex position of specified potential-streamline pairs so that the flownet can be constructed. This strategy greatly simplifies the determination of groundwater flownets, capture zones, solute-front migration, and travel times.

INTRODUCTION

Flownets are a traditional and valuable tool that aid in the analysis and interpretation of groundwater flow and contaminant transport problems. Flownets help to visualize the flowfield, to delineate the capture zone or influence areas of discharge and recharge, to design groundwater development and remediation measures, and to evaluate the effects of different boundary conditions during site characterization. For transport problems, flowpaths and travel times found by particle tracking are used to identify outflow locations and the arrival time of contaminants, as well as to show the advance of a contaminant front within the flow domain.

Usual Methods for Constructing a Flownet

Two methods are usually employed to construct a flownet or flowpath. One method utilizes potential and stream functions to identify the locations of the equipotential and streamlines. Because most numerical procedures provide only the values of the potential and stream function at a limited number of locations, interpolation using a contouring algorithm is usually necessary. Interpolation procedures can introduce additional errors into the flownet geometry delineation.

The second method utilizes knowledge of the velocity field, which is usually deduced from the results of flow modeling. Therefore, the accuracy of the flow paths depends on both the flow model and the interpolation scheme used to construct the velocity field. Apart from the accuracy of the velocity field, the accuracy of calculated travel times is also related to the step size associated with the time-stepping scheme. Fine discretization in both space and time are major constraints to the application of these methods to travel time calculations.

For conditions of steady flow through homogeneous media, traditional real-variable boundary-element methods (RVBEM) and complex-variable boundary-element methods (CVBEM) relax the requirement for discretization of the flow domain. The boundary-element methods may still be appropriate for contaminant transport problems involving long time scales, in which case short-term transients are unimportant and the steady flow assumption is acceptable (Frind and Matanga, 1985). Nonetheless, these methods need to perform either a time-stepping or contour interpolation when plotting flownets and calculating travel times.

Alternate Methods

In this paper we first describe a method for solving flow problems within the physical plane (i.e., solving for unknown boundary potentials) using Ordinary Least Squares (OLS). OLS is used here because it improves the accuracy of the CVBEM solution in the physical plane when the resulting system of algebraic equations is overdetermined. OLS yields a matrix that is symmetric, positive-definite, and diagonally-dominant, which simplifies inversion and yields a better estimate of unknown boundary data. Once the unknown boundary data have been estimated, CVBEM allows the potential and velocities to be calculated at internal points with excellent accuracy and continuity. This capability lays the foundation upon which contaminant transport problems can be addressed.

Instead of finding the value of the complex potential, $w = \phi + i\psi$, for given complex positions, $z = x + iy$, in the physical plane, our method reverses this procedure. We directly physical plane, our method reverses this procedure. We directly identify the locations of flownet intersections for any streamline or equipotential increment. This approach also allows us to calculate travel times more efficiently and accurately. Travel times are calculated, not by time-stepping, but by using an increment in potential.

CVBEM FORMULATION

Two-dimensional, steady, groundwater flow in a homogeneous, isotropic medium is governed by the Laplace equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The principal unknown, u, can be either potential, ϕ, (defined here as the negative product of the total head with the hydraulic conductivity), or stream function, ψ. The boundary value problem is usually solved by numerical methods for arbitrary boundary geometries and for appropriate boundary conditions.

The complex variable boundary element method (CVBEM) provides a method for simultaneously obtaining the solution of the potential and stream function (Hunt and Isaacs, 1981; Hromadka and Lai, 1987). Most CVBEMs employ the Cauchy Integral using complex variables:

$$w[a] = \frac{1}{\theta i} \oint_{\Gamma} \frac{w[z]}{z-a}\, dz \tag{2}$$

where $w[z]=\phi+i\psi$ is the complex potential, $z=x+iy$ is the complex position, i is the imaginary unit, Γ is the boundary of the domain, and θ equals either 2π for a point lying within the domain, or θ equals the inner angle formed by tangential lines before and after the point for points along the boundary.

This differential equation is converted to a boundary integral equation, and is further reduced to an algebraic system of equations after approximation and discretization (Yu and Rasmussen, pending publication):

$$\sum_{k=1}^{NE(M-1)} [a_{jk} - i b_{jk}]\, w_k = 0 \tag{3}$$

which can be written in matrix form as:

$$\underset{(2n \times 2n)}{\begin{bmatrix} A & B \\ -B & A \end{bmatrix}} \underset{(2n \times 1)}{\begin{bmatrix} \Phi \\ \Psi \end{bmatrix}} = \underset{(2n \times 1)}{\begin{bmatrix} 0 \end{bmatrix}} \tag{4}$$

or

$$C \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = 0$$

where

$$C = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \tag{5}$$

It is interesting to note that the coefficients a_{jk} and b_{jk} in Equations (4) and (5) sum to zero in each row, as seen by setting ϕ equal to a constant and setting ψ equal to zero. While the coefficients are also dependent on the form of the shape function selected, and the boundary nodal positions, the coefficients are strikingly similar.

If the boundary geometry is approximated using straight-line segments and linear shape functions are used, the complex potential along each segment can be expressed as:

$$w[z] = \left(\frac{z-z_k}{z_{k+1}-z_k}\right) w_{k+1} + \left(\frac{z_{k+1}-z}{z_{k+1}-z_k}\right) w_k \tag{6}$$

where the subscripts k and $k+1$ are the numberings of the end-nodes in the element, w_k and w_{k+1} are the corresponding nodal values of the complex potential, and z_k and z_{k+1} are the corresponding complex nodal positions.

OLS FORMULATION

The algebraic system of $2n$ equations with $2n$ variables represented above is homogeneous. Also, each equation generally contains $2n$ variables prior to insertion of boundary conditions. In order to make the solution values of ϕ and ψ uniquely determined, one value of ϕ and ψ must be explicitly specified. For a well-posed problem, additional values of ϕ or ψ must be specified along the boundary either directly or indirectly by linking nodal values to neighboring nodes. As a result, the resulting number of unknowns (i.e., m) will always be less than the number of equations ($2n$). Therefore, the system becomes algebraically overdetermined. Our method for solving the overdetermined matrix equations uses the ordinary least squares (OLS) method to determine the "most probable" solution (Lanczos, 1961). The OLS formulation is implemented by first assigning boundary conditions to form the system of equations:

$$C_u \begin{bmatrix} \Phi_u \\ \Psi_u \end{bmatrix} = -\, C_k \begin{bmatrix} \Phi_k \\ \Psi_k \end{bmatrix} \tag{7}$$

where the subscripts u and k refer to unknown and known boundary data, respectively. The matrix C_u has the dimension $(2n \times m)$, where $m < 2n$. The unknown boundary data, w_u, are estimated using (Yu and Rasmussen, pending):

$$w_u = -\,(C_u^T\, C_u)^{-1}\, C_u^T\, C_k\, w_k \tag{8}$$

This approach provides the best estimate of the unknown boundary data in the sense of minimizing the global approximation error when an overdetermined set of matrix equations is present, which is usually the case. It should be noted that the matrix, $C_u^T C_u$, with dimensions of $(m*m)$, is now symmetric, positive definite and diagonally dominant, which is extremely advantageous when solving.

FLOWNET SOLUTION STRATEGY

We note that a complex potential problem in the physical plane, $w[z]$, can be transformed into a complex position problem in the complex potential plane, $z[w]$. The expressions in the physical plane are:

$$\phi = \phi[x,y] \qquad \psi = \psi[x,y] \tag{9}$$

It is relatively straightforward to show:

$$\begin{bmatrix} \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial x}{\partial \psi} \end{bmatrix} = \frac{1}{q^2} \begin{bmatrix} \dfrac{\partial \psi}{\partial y} \\ -\dfrac{\partial \phi}{\partial y} \end{bmatrix} \tag{10}$$

and, similarly:

$$\begin{bmatrix} \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial y}{\partial \psi} \end{bmatrix} = \frac{1}{q^2} \begin{bmatrix} -\dfrac{\partial \psi}{\partial x} \\ \dfrac{\partial \phi}{\partial x} \end{bmatrix} \tag{11}$$

From these equations it follows that:

$$\frac{\partial x}{\partial \phi} = \frac{\partial y}{\partial \psi} = \frac{q_x}{q^2} \qquad \frac{\partial y}{\partial \phi} = -\frac{\partial x}{\partial \psi} = \frac{q_y}{q^2} \tag{12}$$

and: $$\frac{\partial^2 x}{\partial \phi^2} + \frac{\partial^2 x}{\partial \psi^2} = 0 \qquad \frac{\partial^2 y}{\partial \phi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0 \tag{13}$$

These equations are, respectively, the equivalent Cauchy-Riemann conditions and governing equations in the complex potential plane. A fundamental difficulty associated with solving these latter equations in the complex potential plane is the definition of complex position boundary conditions. This problem is bypassed by solving for unknown potentials in the physical plane using the OLS solution method described previously. It is advantageous to use linear interpolators for both the complex position and complex potential problems because the mutual linearity of $w[z]$ with $z[w]$ is assured. This formulation allows the use of the same discretization equation by interchanging w with z. This allows us to find the complex position for a specified complex potential within a flow domain, which is especially useful for finding the unknown position of a boundary node, such as along a free surface (Liggett and Liu, 1983).

EXAMPLES

As an example of the use of this strategy, Figure 1 presents the flownet solution for groundwater flow beneath a dam containing a sheet pile (Fetter, 1988). Due to symmetry, only the right half of the flow problem is presented. A prescribed head is assigned on the upstream and downstream surfaces. No-flow boundaries are prescribed along the right and bottom sides, and along the dam foundation.

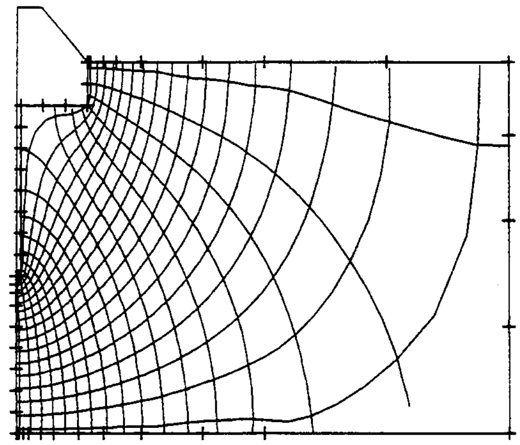

Figure 1. Flow of water under a dam with a sheet pile.

The dipole problem can be used to demonstrate a source and a sink, each with a unit strength (Figure 2). The source and sink are separated by a distance of two units in no ambient flow. One reason for choosing such an example is the availability of an exact analytical solution for the travel time (Muskat, 1946). For Dupuit flow conditions, we use the discharge potential and the discharge streamline function to replace the potential and stream functions, respectively (Strack, 1989).

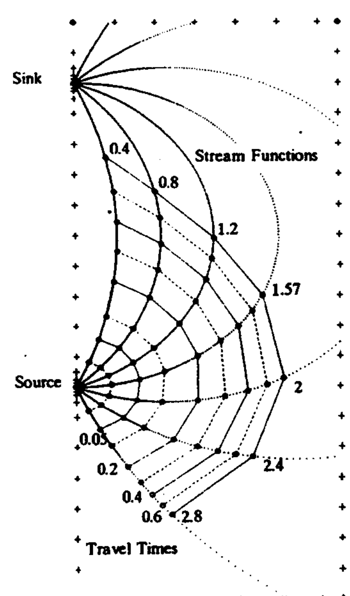

Figure 2. Dipole streamlines and solute fronts.

Another demonstration of the strategy is a regional flow problem (Tóth, 1963). Figure 3 presents a regional flow problem using an elevation view of the flownet in the physical plane (a), and the calculated boundary potentials in the complex potential plane (b). The upper surface of the complex position consists of a specified head equal to the elevation, while the remaining surfaces are noflow boundaries. The single-valued property of x and y in the complex potential plane is lost in the area of the intervening valley because of folding. It should be noted that subdomains meeting the one-to-one correspondence must be located before finding the position of the specified potential in this part of the domain.

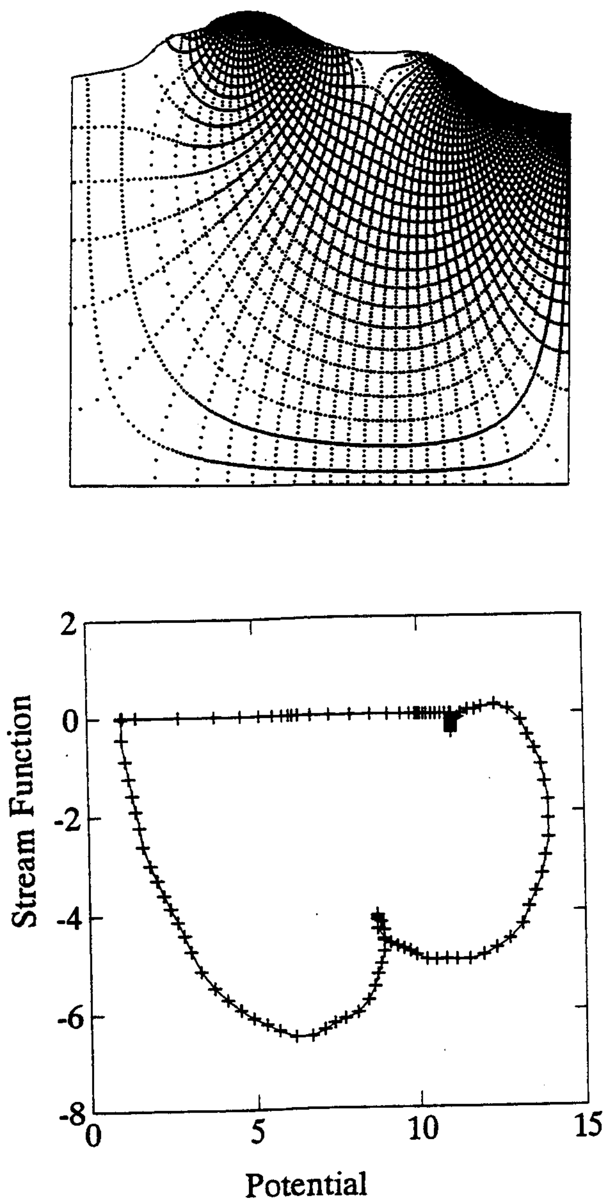

Figure 3. Regional ground-water flow.

To resolve the problem of overlapping positions in the complex potential plane, we can subdivide the physical flow domain into separate flow subdomains using a dividing streamline to maintain the single-valued property of x and y in their images, respectively. The position of the streamline along the flow divide is found by starting at the location on the boundary where it originates or terminates. We employ a first-order approximation of the potential increase, $\Delta\phi$ and $\Delta\psi$, along the divide. One advantage of this approach is that errors in the position of successive points along the divide are never accumulated. Also, the procedure can proceed either upstream or downstream. When the dividing streamline includes a stagnation point, care must be taken to initiate the procedure slightly away from the point to avoid numerical difficulties. The method can be used for locating an equipotential line.

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